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The crumpling transition of membranes driven by quantum fluctuations in a $D = \epsilon$ expansion

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Abstract

We consider a D-dimensional fluid membrane in a $(D + 1)$-dimensional embedding space, subject to quantum fluctuations. The corresponding action is invariant under coordinate transformations and depends only on the shape of the membrane and its variation, neglecting tangential degrees of freedom. We calculate the resulting field theory to one loop order in a $D = \epsilon$ expansion and find a quantum transition even at $T = 0$.

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1. Introduction

Fluid membranes, like biomembranes, belong to the classical world—there is no need to take quantum fluctuations into consideration since typical temperatures are high and sizes are large. Nevertheless, we might reduce temperature down to zero and increase $\hbar$ in a *Gedankenexperiment* and study the quantum fluctuations of a (non-relativistic) flexible membrane. The most important feature of the resulting model is a second-order quantum transition at a finite $\hbar$ from an almost flat to a crumpled phase, contrary to the thermal case, where the membrane remains always in the crumpled phase [1]. Ingredients of the action describing the quantum membrane are a kinetic energy term, surface tension and the Canham–Helfrich curvature energy [2]. The latter elastic term disfavours curved configuration of the membrane and is proportional to the square of the mean curvature, integrated over the surface of the membrane. The action should not depend on the internal coordinates which are used to parametrize the surface—it should be a functional of the geometry of the membrane only.

Our model serves as a toy model for quantum interfaces like the helium liquid–vapour interface at very low temperatures [3, 4]. It is remarkable that He^3 enriches at a He^4 interface and acts as a surfactant, lowering the surface tension of the He^4 interface [4]. Therefore, the next-leading curvature terms become relevant. The He^3 film on top of the He^4 bulk could be in fact a suitable candidate for the quantum membrane under investigation. It is, however, unlikely that the predicted second-order phase transition from a flat to a crumpled interface

(at finite $\overline{h}$) can be observed experimentally. The surface tension of the He-system is still non-zero, and moreover, there is no direct way to change $\overline{h}$ in an experiment. Nevertheless, it might be possible by changing the He^3 concentration to tune the system closer to the transition point.

So far, non-relativistic quantum membranes with curvature stiffness and surface tension were discussed in [5] at fixed dimension $D = 2$ of the membrane and $d = 3$ of the embedding space and in [6] for an infinite dimensional embedding space $d \to \infty$. Extending these results we study the quantum fluctuations of a membrane in a systematic $D = \epsilon$ expansion and find a fixed point at zero temperature and finite $\overline{h}$ (for $D > 0$), which is not seen by [5].

2. The model

Following [5] we start from the (imaginary time) action

$$S_0 = \int \mathrm{d}t\, \mathrm{d}^D\sigma \sqrt{g} \left(\frac{1}{2\nu}(\partial_t \boldsymbol{X})^2 + r + \frac{2}{\alpha} H^2 \right) \tag{2.1}$$

where $\boldsymbol{X}(\sigma, t)$ describes the time-dependent D-dimensional surface embedded in a $(D+1)$-dimensional Euclidean space. Hence σ is a D-dimensional set of internal coordinates which parametrize the surface. $\mathrm{d}^D\sigma\sqrt{g}$ is the invariant element of area, H is the mean curvature, $1/\nu$ is the mass density, r is the surface tension and $1/\alpha$ is the bending rigidity. The action (2.1) only makes sense in Lagrangian coordinates, i.e. coordinates which follow the elements of fluid. Then, $\boldsymbol{X}(\sigma, t)$ is (for fixed σ) the trajectory of an element of fluid and $\partial_t \boldsymbol{X}$ the corresponding velocity. The velocity might be decomposed into a normal part $v_\perp = \boldsymbol{N} \cdot \partial_t \boldsymbol{X}$ and the tangential parts $u_i = \partial_i \boldsymbol{X} \cdot \partial_t \boldsymbol{X}$, where $\boldsymbol{N}$ is the normal vector and $\partial_i \boldsymbol{X} = \partial \boldsymbol{X}/\partial\sigma^i$ are the tangential vectors (see e.g. [7, 8]). The normal velocity $v_\perp$ allows for a geometrical interpretation—it encodes the variations of the shape of the membrane, whereas the tangential velocity simply generates reparametrizations of the surface (coordinate transformations). Indeed, a purely tangential velocity field leaves the shape of the membrane unchanged and, therefore, belongs to degrees of freedom beyond the geometric ones[1].

Instead of using action (2.1) and taking care of the tangential degrees of freedom, we write down a simplified action, depending only on the shape of the membrane and its variation (with conveniently chosen coupling constants)

$$S_0/\overline{h} = \int \mathrm{d}t\, \mathrm{d}^D\sigma \sqrt{g} \left(\frac{\lambda}{2\overline{h}} v_\perp^2 + \frac{r}{\overline{h}} + \frac{2}{\overline{h}\lambda} H^2 \right). \tag{2.2}$$

It can be seen easily that $v_\perp$ is really a scalar under general *time-dependent* coordinate transformations $\sigma \to \sigma(\sigma', t)$. Consequently, the action S_0 itself is invariant under any time-dependent coordinate transformation. To calculate the partition function $\mathcal{Z} = \int D[\boldsymbol{X}] \exp(-S_0/\overline{h})$ and related expectation values, we have to sum over all physically distinct surfaces $\boldsymbol{X}(\sigma, t)$ in a reparametrization-invariant manner. Being far from trivial (see [9], the quantum case does not pose an extra complication), we have to restrict the discussion of the measure problem to few remarks. At first, we have to choose a certain representation of the surfaces (gauge fixing) in order to avoid over-counting of surfaces with identical shapes, but different coordinate systems. A common and practical choice is a representation of the surface in terms of a (time dependent) height variable $f(\mathbf{x}, t)$—the Monge representation ($\mathbf{x}$ is a D-dimensional Euclidean vector)

$$\boldsymbol{X}(\mathbf{x}, t) = (\mathbf{x},\, f(\mathbf{x}, t)) \tag{2.3}$$

[1] The divergence of the tangential velocity field, however, is fixed by the (geometrical) condition $2Hv_\perp = D^i u_i$ in case of an incompressible membrane.

which is connected to the Lagrangian coordinates by a particular time-dependent coordinate transformation. We convert the action (2.2) into the Monge representation using $v_\perp = \partial_t f/\sqrt{g}$ (where $g = 1 + (\nabla f)^2$) and obtain

$$\begin{aligned} S_0/\overline{h} &= \int \mathrm{d}t\, \mathrm{d}^D x \left[\frac{\lambda}{2\overline{h}} \frac{(\partial_t f)^2}{\sqrt{g}} + \frac{r}{\overline{h}}\sqrt{g} + \frac{1}{2\overline{h}\lambda}\sqrt{g}\left(\nabla \cdot \left(\frac{1}{\sqrt{g}}\nabla f\right)\right)^2\right] \\ &= \int \mathrm{d}t\, \mathrm{d}^D x \left[\frac{\lambda}{2\overline{h}}(\partial_t f)^2 + \frac{r}{2\overline{h}}\partial_i f \partial_i f + \frac{1}{2\overline{h}\lambda}(\partial^2 f)^2 \right. \\ &\quad - \frac{\lambda}{4\overline{h}}(\partial_t f)^2 \partial_i f \partial_i f - \frac{r}{8\overline{h}}\partial_i f \partial_i f \partial_j f \partial_j f \\ &\quad \left. - \frac{1}{4\overline{h}\lambda}\partial_i f \partial_i f (\partial^2 f)^2 - \frac{1}{\overline{h}\lambda}\partial_i f \partial_j f\, \partial_i \partial_j f\, \partial^2 f \right] + \mathrm{O}(f^6) \end{aligned} \tag{2.4}$$

where $i, j = 1, \ldots, D$.[2] The corresponding invariant measure $D[f]$ differs from the naive measure $\propto \prod_{\mathbf{x}} \int \mathrm{d}f(\mathbf{x})$ by the so called Fadeev–Popov determinant and the Liouville term, which, however, contribute to two-loop and higher orders only. To one-loop order, we may safely use the naive measure instead [9]. The lower critical dimension of the theory is $D = 0$, where the coupling constants in front of the kinetic energy and in front of the curvature energy become marginal, as can be seen from the dimensions of the coupling constants (L = length), which are $\overline{h} \sim L^D$ and $r \sim L^{-2}$ (λ is rendered dimensionless, $t \sim L^2$). Therefore, we have to calculate the quantum fluctuations of (2.4) in a double $\overline{h}$ and $D = \epsilon$ expansion, which is done here to one-loop order with the help of dimensional regularization and the minimal subtraction scheme in analogy with [10]. The surface tension r is a relevant parameter of the theory and is zero right at the critical point. In fact, a non-zero r imposes a finite correlation length $\xi = r^{-1/2}$ on the propagator of action (2.4). Not included in the action (2.4) are the integral over the scalar curvature R (which does not yield a contribution to one-loop order and which is a topological invariant for $D = 2$) and a boundary term—the cross term $2\int \mathrm{d}t \mathrm{d}^D x \sqrt{g}\, v_\perp H = -\int \mathrm{d}t \partial_t A$, where A is the surface area.

3. Field theory

The bare $T = 0$ two-point vertex function $\Gamma_{0,2}(q, \omega)$ reads (denoting from now on bare quantities with a subscript 0)

$$\begin{aligned} \Gamma_{0,2}(q,\omega) = {} & \frac{\lambda_0}{\overline{h}_0}\omega^2 + \frac{r_0}{\overline{h}_0}q^2 + \frac{1}{\overline{h}_0\lambda_0}q^4 + \frac{\omega^2}{2}\lambda_0(r_0\lambda_0)^{\epsilon/2}I_\epsilon - \frac{q^2}{2+\epsilon}r_0(r_0\lambda_0)^{\epsilon/2}I_\epsilon \\ & + q^4\left(\frac{1}{2}+\frac{2}{\epsilon}\right)\frac{1}{\lambda_0}(r_0\lambda_0)^{\epsilon/2}I_\epsilon \end{aligned} \tag{3.1}$$

where $I_\epsilon = (4\pi)^{-\epsilon/2-1/2}\Gamma(1-\epsilon/2)\Gamma(\epsilon/2+1/2)/\,\Gamma(1+\epsilon/2)$ (I_ϵ is finite in the limit $\epsilon \to 0$). We introduce renormalized couplings $\overline{h}, \lambda, r$ by $\overline{h}_0 = \mu^{-\epsilon}\overline{h}Z_{\overline{h}}/I_\epsilon$, $r_0 = Z_r r$ and $\lambda_0 = Z_\lambda \lambda$

[2] The action (2.4) differs from the one used in [5] by a wrong sign in front of the first vertex. The wrong sign, however, does not affect any consequence drawn by [5] (for zero temperature), since the authors study the field theory not at the lower critical dimension, but at the dimension $D = 2$, where the flow of the coupling constants is mainly determined by their naive dimensions.

(μ is an arbitrary momentum scale) and require the renormalized vertex function

$$\Gamma_2(q,\omega) = \frac{\mu^\epsilon I_\epsilon}{\bar{h}}\left(\frac{Z_\lambda}{Z_{\bar{h}}}\lambda\omega^2 + \frac{Z_r}{Z_{\bar{h}}}rq^2 + \frac{1}{\lambda Z_\lambda Z_{\bar{h}}}q^4 + \frac{\omega^2}{2}\lambda\bar{h}Z_\lambda - \frac{q^2}{2+\epsilon}r\bar{h}Z_r + q^4\left(\frac{1}{2}+\frac{2}{\epsilon}\right)\frac{\bar{h}}{\lambda Z_\lambda}\right) \tag{3.2}$$

to be finite in the limit $\epsilon \to 0$. We find $Z_{\bar{h}} = Z_\lambda = Z_r = 1 + \bar{h}/\epsilon + \mathrm{O}(\bar{h}^2)$. The RG-equation is readily derived from the fact that the bare quantities do not depend on the momentum scale μ, yielding

$$(\mu\partial_\mu + \beta(\bar{h})\partial_{\bar{h}} + \gamma r\partial_r + \zeta\lambda\partial\lambda)\Gamma_2(q,\omega) = 0 \tag{3.3}$$

with the beta-function $\beta(\bar{h}) = \bar{h}(\epsilon - \bar{h})$. The theory has an ultraviolet stable fixed point (not seen by [5]) $\bar{h}^* = \epsilon$ which corresponds to a quantum-phase transition at a *finite* $\bar{h}$ (for $D > 0$ and $T = 0$) from a smooth phase for small $\bar{h}$ to a crumpled phase for $\bar{h} > \bar{h}^*$. The main effect of quantum fluctuations on the statistics of membranes is a shift of the lower critical dimension from $D = 2$ to $D = 0$. For slightly larger dimensions a quantum-phase transition takes place at a finite $\bar{h}^*$. We expect that this picture remains true up to the physical dimension $D = 2$, although the numerical accuracy for the one-loop critical exponents (not presented in this short communication) should be quite limited.

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Appendix A. Details of the calculation

The bare two-point vertex function reads

$$\Gamma_{0,2}(q,\omega) = \frac{\lambda}{\bar{h}}\omega^2 + \frac{r}{\bar{h}}q^2 + \frac{1}{\bar{h}\lambda}q^4 - \frac{\lambda}{2\bar{h}}\omega^2 I_2 - \frac{\lambda}{2\bar{h}}q^2 I_1 - \frac{r}{2\bar{h}}q^2 I_2 - \frac{r}{\bar{h}\epsilon}q^2 I_2 - \frac{1}{2\bar{h}\lambda}q^2 I_3 - \frac{1}{2\bar{h}\lambda}q^4 I_2 - \frac{2}{\bar{h}\lambda\epsilon}q^2 I_3 - \frac{2}{\bar{h}\lambda\epsilon}q^4 I_2 \tag{A.1}$$

with the Feynman-integrals

$$I_1 = \bar{h}\int\frac{\mathrm{d}\omega}{2\pi}\frac{\mathrm{d}^\epsilon q}{(2\pi)^\epsilon}\frac{\omega^2}{\lambda\omega^2 + rq^2 + \lambda^{-1}q^4}$$
$$I_2 = \bar{h}\int\frac{\mathrm{d}\omega}{2\pi}\frac{\mathrm{d}^\epsilon q}{(2\pi)^\epsilon}\frac{q^2}{\lambda\omega^2 + rq^2 + \lambda^{-1}q^4} \tag{A.2}$$
$$I_3 = \bar{h}\int\frac{\mathrm{d}\omega}{2\pi}\frac{\mathrm{d}^\epsilon q}{(2\pi)^\epsilon}\frac{q^4}{\lambda\omega^2 + rq^2 + \lambda^{-1}q^4}.$$

Within dimensional regularization we have $\left(\int \mathrm{d}\omega\, \mathrm{d}^\epsilon q \equiv 0\right)$

$$I_1 = \frac{\bar{h}}{\lambda}\int\frac{\mathrm{d}\omega}{2\pi}\frac{\mathrm{d}^\epsilon q}{(2\pi)^\epsilon}\frac{\lambda\omega^2 + rq^2 + \lambda^{-1}q^4 - rq^2 - \lambda^{-1}q^4}{\lambda\omega^2 + rq^2 + \lambda^{-1}q^4}$$
$$= -\frac{r}{\lambda}I_2 - \frac{1}{\lambda^2}I_3. \tag{A.3}$$

To evaluate I_2, we substitute $\omega \to \omega q^2$ and find

$$I_2 = \bar{h} \int \frac{\mathrm{d}^\epsilon q}{(2\pi)^\epsilon} \frac{\mathrm{d}\omega}{2\pi} \frac{q^2}{\lambda\omega^2 q^2 + r + \lambda^{-1} q^2}. \tag{A.4}$$

Now we are able to perform the q-integration

$$I_2 = -\frac{\bar{h}}{r} \int \frac{\mathrm{d}\omega}{2\pi} \left(\frac{r}{\lambda\omega^2 + \lambda^{-1}} \right)^{1+\epsilon/2} I \tag{A.5}$$

where

$$I = \int \frac{\mathrm{d}^\epsilon k}{(2\pi)^\epsilon} \frac{-k^2}{k^2 + 1} = \frac{1}{(4\pi)^{\epsilon/2}} \Gamma(1 - \epsilon/2). \tag{A.6}$$

With the help of $(2\pi)^{-1} \int \mathrm{d}s (s^2 + 1)^{-1-\epsilon/2} = (4\pi)^{-1/2} \Gamma(\epsilon/2 + 1/2) / \Gamma(1 + \epsilon/2)$ we obtain

$$I_2 = -\bar{h} (r\lambda)^{\epsilon/2} I_\epsilon \tag{A.7}$$

where

$$I_\epsilon = \frac{1}{(4\pi)^{\epsilon/2+1/2}} \frac{\Gamma(1 - \epsilon/2)\Gamma(\epsilon/2 + 1/2)}{\Gamma(1 + \epsilon/2)}. \tag{A.8}$$

An analogous calculation yields

$$I_3 = -\lambda r \frac{1 + \epsilon}{2 + \epsilon} I_2 = \bar{h} \lambda r (r\lambda)^{\epsilon/2} \frac{1 + \epsilon}{2 + \epsilon} I_\epsilon. \tag{A.9}$$

Within a cut-off regularization scheme instead of dimensional regularization additional divergent terms show up in equation (A.3) and equation (A.9) which can be absorbed by an additive renormalization of the surface tension r.

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